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Supplement to Third Quarterly Report  
OPTIMUM APERTURE STUDY

Technical Documentary Report No. RADC-TDR-63-10, Suppl 1  
May 1963

ROME AIR DEVELOPMENT CENTER  
Research and Technology Division  
Air Force Systems Command  
United States Air Force  
Griffiss Air Force Base  
New York



Project No. 4506, Task No. 450604

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(Prepared under Contract No. AF30(602)-2676  
by D. Lee, Electronic Systems and Products  
Division, Martin Company, Baltimore 3, Md.)

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## ABSTRACT

The object of this contract is to study the applicability of the Wiener-Spencer Theorem to antennas. This theorem states that minimum standard deviation of the far-field pattern occurs when the illumination function corresponds to the lowest mode of vibration of a membrane stretched across the aperture opening.

This report presents the investigation of four selected nonoptimum illuminations for the elliptical apertures. Approximations are used to obtain expressions for far-field power patterns, and second moments are tabulated. In addition, illuminations and far-field power patterns are plotted.

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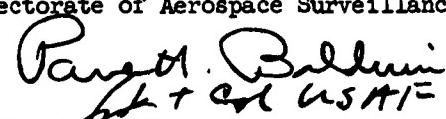
## PUBLICATION REVIEW

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## I. INTRODUCTION

The Third Quarterly Report states that a second group of nonoptimum illuminations for elliptical apertures of the form

$$F(\xi, \eta) = \left[ (1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0} \right]^N \quad N = 1, 2, 3, 4$$

will be investigated. It further states that a comparison will be made between the optimum and nonoptimum illuminations. The following work has been accomplished:

- (1) The far-field power patterns of elliptical apertures with four selected nonoptimum illuminations were derived through approximation.
- (2) An IBM 1620 program was written to tabulate the moments.
- (3) Investigation was made between optimum and nonoptimum illuminations to the degree of improvement in terms of the second moments, the side lobes and the beamwidth.
- (4) Far-field power patterns of four selected nonoptimum illuminations were plotted along major axes.

## II. ELLIPTICAL APERTURE WITH NONOPTIMUM ILLUMINATION

The far-field voltage power pattern of elliptical aperture is given by

$$G(u, v) = \int \int e^{i(ux + vy)} F dx dy.$$

For the optimum case,  $F$  is a product of two Mathieu Functions

$$[Ce(q, \xi)] [ce(q, \eta)]$$

In elliptical coordinates,

$$G(u, v) = \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} e^{ih[u \cosh \xi \cos \eta + v \sinh \xi \sin \eta]} (\cosh 2\xi - \cos 2\eta) F(\xi, \eta) d\xi d\eta$$

where  $F(\xi, \eta)$  is the illumination distribution.

The zeroth moment is given by

$$\mu_0 = \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} F^2(\xi, \eta) (\cosh 2\xi - \cos 2\eta) d\xi d\eta,$$

and the second moment is given by

$$\mu_2 = \int_0^{2\pi} \int_0^{\xi_0} \left[ \left( \frac{\partial F}{\partial \xi} \right)^2 + \left( \frac{\partial F}{\partial \eta} \right)^2 \right] d\xi d\eta.$$

For a nonoptimum illumination, let

$$F(\xi, \eta) = \left[ (1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0} \right]^N$$

The illumination satisfies the conditions

$$F(\xi, \eta) = F(\xi, \eta + 2\pi)$$

$$F(\xi_0, \eta) = 0.$$

Thus,

$$\begin{aligned} G(u, v) &= \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} e^{ih} [u \cosh \xi \cos \eta + v \sinh \xi \sin \eta] (\cosh 2\xi \\ &\quad - \cos 2\eta) \left[ (1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0} \right]^N d\xi d\eta \\ &= \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} e^{ih} [u \cosh \xi \cos \eta + v \sinh \xi \sin \eta] \cosh 2\xi \left[ (1 \right. \\ &\quad \left. + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0} \right]^N d\xi d\eta \\ &\quad - \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} e^{ih} [u \cosh \xi \cos \eta + v \sinh \xi \sin \eta] \cos 2\eta \left[ (1 \right. \\ &\quad \left. + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0} \right]^N d\xi d\eta. \end{aligned}$$

The preceding integrals do not appear to be solvable in closed form. Instead, we examine  $G(u, 0)$ .

$$\begin{aligned} G(u, 0) &= \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} e^{ih} u \cosh \xi \cos \eta \cosh 2\xi \left[ (1 \right. \\ &\quad \left. + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0} \right]^N d\xi d\eta \\ &\quad - \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} e^{ih} u \cosh \xi \cos \eta \cos 2\eta \left[ (1 \right. \\ &\quad \left. + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0} \right]^N d\xi d\eta. \end{aligned}$$

For the given aperture  $\xi_0 = 0.277$ ,  $0 \leq \xi \leq \xi_0$

$$\cos \xi \sim 1$$

$$\sinh \xi \sim \xi.$$

Thus,

$$G(u, 0) = \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} e^{ih u \cos \eta} \cosh 2\xi \left[ (1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0} \right]^N d\xi d\eta$$

$$- \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} e^{ih u \cos \eta} \cos 2\eta \left[ (1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0} \right]^N d\xi d\eta.$$

Recall that

$$e^{ix \cos \theta} = J_0(x) + 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(x) \cos 2K \theta$$

$$+ 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(x) \cos (2K-1) \theta.$$

Thus,

$$G(u, 0) = \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} \left[ J_0(hu) + 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(hu) \cos 2K \eta \right. \\ \left. + 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos (2K-1) \eta \right] \cosh 2\xi \left[ (1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0} \right]^N d\xi d\eta$$

$$\begin{aligned}
 & -\frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} \left[ J_0(hu) + 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(hu) \cos 2K \eta \right. \\
 & \left. + 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos(2K-1)\eta \right] \cos 2\eta \left[ (1 \right. \\
 & \left. + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0} \right]^N d\xi d\eta.
 \end{aligned}$$

For N = 1,

$$\begin{aligned}
 G_1(u, 0) = & \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} \left[ J_0(hu) + 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(hu) \cos 2K \eta \right. \\
 & \left. + 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos(2K-1)\eta \right] \cosh 2\xi_0 \left[ (1 \right. \\
 & \left. + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0} \right] d\xi d\eta \\
 & - \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} \left[ J_0(hu) + 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(hu) \cos 2K \eta \right. \\
 & \left. + 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos(2K-1)\eta \right] \cos 2\eta \left[ (1 \right. \\
 & \left. + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0} \right] d\xi d\eta
 \end{aligned}$$

$$\begin{aligned}
&= \frac{h^2}{2} \left( \int_0^{2\pi} J_0(hu) (1 + a \sin^2 \eta) d\eta \right) \left( \int_0^{\xi_0} \cosh 2\xi \cos \frac{\pi \xi}{2\xi_0} d\xi \right) \\
&\quad + \frac{h^2}{2} \left( \int_0^{2\pi} 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(hu) \cos 2K \eta (1 \right. \\
&\quad \left. + a \sin^2 \eta) d\eta \right) \left( \int_0^{\xi_0} \cosh 2\xi \cos \frac{\pi \xi}{2\xi_0} d\xi \right) \\
&\quad + \frac{h^2}{2} \left( \int_0^{2\pi} 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos (2K-1) \eta (1 \right. \\
&\quad \left. + a \sin^2 \eta) d\eta \right) \left( \int_0^{\xi_0} \cosh 2\xi \cos \frac{\pi \xi}{2\xi_0} d\xi \right) \\
&\quad - \frac{h^2}{2} \left( \int_0^{2\pi} J_0(hu) \cos 2\eta [ (1 + a \sin^2 \eta) d\eta \right) \left( \int_0^{\xi_0} \cos \frac{\pi \xi}{2\xi_0} d\xi \right) \\
&\quad - \frac{h^2}{2} \left( \int_0^{2\pi} 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(hu) \cos 2K \eta \cos 2\eta (1 \right. \\
&\quad \left. + a \sin^2 \eta d\eta \right) \left( \int_0^{\xi_0} \cos \frac{\pi \xi}{2\xi_0} d\xi \right) \\
&\quad - \frac{h^2}{2} \left( \int_0^{2\pi} 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos (2K-1) \eta \cos 2\eta (1
\end{aligned}$$

$$+ a \sin^2 \eta) d\eta \Bigg) \left( \int_0^{\xi_0} \cos \frac{\pi \xi}{2\xi_0} d\xi \right).$$

Here,

$$\int_0^{\xi_0} \cosh 2\xi \cos \frac{\pi \xi}{2\xi_0} d\xi = \frac{2\xi_0 \pi \cosh 2\xi_0}{16 \xi_0^2 + \pi^2}$$

$$\int_0^{2\pi} J_0(hu) (1 + a \sin^2 \eta) d\eta = 2\pi \left(1 + \frac{a}{2}\right) J_0(hu)$$

$$\int_0^{2\pi} 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(hu) \cos 2K \eta (1 + a \sin^2 \eta) d\eta = a \pi J_2(hu)$$

$$\int_0^{2\pi} 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos (2K-1) \eta (1 + a \sin^2 \eta) d\eta = 0$$

$$\int_0^{\xi_0} \cos \frac{\pi \xi}{2\xi_0} d\xi = \frac{2\xi_0}{\pi}$$

$$\int_0^{2\pi} J_0(hu) \cos 2\eta [(1 + a \sin^2 \eta)] d\eta = -\frac{a}{2} \pi J_0(hu)$$

$$\begin{aligned} & \int_0^{2\pi} 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(hu) \cos 2K \eta \cos 2\eta [(1 + a \sin^2 \eta)] d\eta \\ &= -2\pi \left(1 + \frac{a}{2}\right) J_2(hu) - \frac{a}{2} J_4(hu) \end{aligned}$$

$$\int_0^{2\pi} 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos(2K-1)\eta \cos 2\eta \left[ (1 + a \sin^2 \eta) \right] d\eta = 0.$$

Therefore,

$$G_1(u, 0) = h^2 \xi_0 \left[ \left\{ \frac{a}{2} + \left(1 + \frac{a}{2}\right) \frac{2\pi^2 \cosh 2\xi_0}{16 \xi_0^2 + \pi^2} \right\} J_0(hu) + \left\{ 2 \left(1 + \frac{a}{2}\right) + \frac{a\pi^2 \cosh 2\xi_0}{16 \xi_0^2 + \pi^2} \right\} J_2(hu) + \frac{a}{2} J_4(hu) \right]$$

For N = 2,

$$G_2(u, 0) = \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} \left[ J_0(hu) + 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(hu) \cos 2K\eta + 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos(2K-1)\eta \right] \cdot \cosh 2\xi \left[ (1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0} \right]^2 d\xi d\eta - \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} \left[ J_0(hu) + 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(hu) \cos 2K\eta + 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos(2K-1)\eta \right] \cdot \cos 2\eta \left[ (1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0} \right]^2 d\xi d\eta$$

$$\begin{aligned}
& + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0} \Big]^2 d\xi d\eta \\
& = \frac{h^2}{2} \left( \int_0^{2\pi} J_0(hu) (1 + a \sin^2 \eta)^2 d\eta \right) \left( \int_0^{\xi_0} \cosh 2\xi \cos^2 \frac{\pi \xi}{2\xi_0} d\xi \right) \\
& + \frac{h^2}{2} \left( \int_0^{2\pi} 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(hu) \cos 2K \eta (1 \right. \\
& \quad \left. + a \sin^2 \eta)^2 d\eta \right) \left( \int_0^{\xi_0} \cosh 2\xi \cos^2 \frac{\pi \xi}{2\xi_0} d\xi \right) \\
& + \frac{h^2}{2} \left( \int_0^{2\pi} 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos (2K-1) \eta (1 \right. \\
& \quad \left. + a \sin^2 \eta)^2 d\eta \right) \left( \int_0^{\xi_0} \cosh 2\xi \cos^2 \frac{\pi \xi}{2\xi_0} d\xi \right) \\
& - \frac{h^2}{2} \left( \int_0^{2\pi} J_0(hu) \cos 2\eta [(1 + a \sin^2 \eta)]^2 d\eta \right) \left( \int_0^{\xi_0} \cos^2 \frac{\pi \xi}{2\xi_0} d\xi \right) \\
& - \frac{h^2}{2} \left( \int_0^{2\pi} 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(hu) \cos 2K \eta \cos 2\eta \right. \\
& \quad \left. [(1 + a \sin^2 \eta)]^2 d\eta \right) \left( \int_0^{\xi_0} \cos^2 \frac{\pi \xi}{2\xi_0} d\xi \right)
\end{aligned}$$

$$\begin{aligned}
 & -\frac{h^2}{2} \left( \int_0^{2\pi} 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos(2K-1)\eta \cos 2\eta \right. \\
 & \left. + a \sin^2 \eta \right]^2 d\eta \left( \int_0^{\xi_0} \cos^2 \frac{\pi \xi}{2\xi_0} d\xi \right).
 \end{aligned}$$

Here,

$$\int_0^{\xi_0} \cosh 2\xi \cos^2 \frac{\pi \xi}{2\xi_0} d\xi = \frac{\pi^2 \sinh 2\xi_0}{4(4\xi_0^2 + \pi^2)}$$

$$\int_0^{2\pi} J_0(hu) (1 + a \sin^2 \eta)^2 d\eta = 2\pi \left( 1 + a + \frac{3}{8} a^2 \right) J_0(hu)$$

$$\begin{aligned}
 & \int_0^{2\pi} 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(hu) \cos 2K\eta (1 + a \sin^2 \eta)^2 d\eta = 2\pi a \left( 1 \right. \\
 & \left. + \frac{a}{2} \right) J_2(hu) + \frac{a^2}{4} \pi J_4(hu)
 \end{aligned}$$

$$\int_0^{2\pi} 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos(2K-1)\eta (1 + a \sin^2 \eta)^2 d\eta = 0$$

$$\int_0^{\xi_0} \cos^2 \frac{\pi \xi}{2\xi_0} d\xi = \frac{\xi_0}{2}$$

$$\int_0^{2\pi} J_0(hu) \cos 2\eta (1 + a \sin^2 \eta)^2 d\eta = -a \left( 1 + \frac{a}{2} \right) \pi J_0(hu)$$

$$\begin{aligned}
& \int_0^{2\pi} 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(hu) \cos 2K \eta \cos 2\eta (1 + a \sin^2 \eta)^2 d\eta \\
& = - \left( 2 + 2a + \frac{7}{8} a^2 \right) \pi J_2(hu) - a \left( 1 + \frac{a}{2} \right) \pi J_4(hu) - \frac{a^2}{8} \pi J_6(hu) \\
& \int_0^{2\pi} 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos (2K-1) \eta \cos 2\eta (1 \\
& \quad + a \sin^2 \eta)^2 d\eta = 0.
\end{aligned}$$

Therefore,

$$\begin{aligned}
G_2(u, 0) &= h^2 \pi \left[ \left\{ \left( 1 + a + \frac{3}{8} a^2 \right) \frac{\pi^2 \sinh 2\xi_0}{4(4\xi_0^2 + \pi^2)} + \frac{\xi_0 a}{4} \left( 1 \right. \right. \right. \\
&\quad \left. \left. \left. + \frac{a}{2} \right) \right\} J_0(hu) + \left\{ a \left( 1 + \frac{a}{2} \right) \frac{\pi^2 \sinh 2\xi_0}{4(4\xi_0^2 + \pi^2)} + \frac{\xi_0}{4} \left( 2 \right. \right. \\
&\quad \left. \left. + 2a + \frac{7}{8} a^2 \right) \right\} J_2(hu) + \left\{ \frac{a^2}{8} \cdot \frac{\pi^2 \sinh 2\xi_0}{4(4\xi_0^2 + \pi^2)} \right. \\
&\quad \left. \left. + \frac{\xi_0 a}{4} \left( 1 + \frac{a}{2} \right) \right\} J_4(hu) + \frac{\xi_0 a^2}{32} J_6(hu) \right].
\end{aligned}$$

For N = 3,

$$\begin{aligned}
G_3(u, 0) &= \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} \left[ J_0(hu) + 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(hu) \cos 2K \eta \right. \\
&\quad \left. + 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos (2K-1) \eta \right] \cosh 2\xi \left[ \left( 1 \right. \right. \\
&\quad \left. \left. + a \sin^2 \eta \right) \cos \frac{\pi \xi}{2\xi_0} \right]^3 d\xi d\eta
\end{aligned}$$

$$\begin{aligned}
& - \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} \left[ J_0(hu) + 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(hu) \cos 2K\eta \right. \\
& \quad \left. + 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos(2K-1)\eta \right] \cdot \cos 2\eta \left[ (1 \right. \\
& \quad \left. + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0} \right]^3 d\xi d\eta \\
& = \frac{h^2}{2} \left( \int_0^{2\pi} J_0(hu) (1 + a \sin^2 \eta)^3 d\eta \right) \left( \int_0^{\xi_0} \cosh 2\xi \cos^3 \left( \frac{\pi \xi}{2\xi_0} \right) d\xi \right) \\
& \quad + \frac{h^2}{2} \left( \int_0^{2\pi} 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(hu) \cos 2K\eta (1 \right. \\
& \quad \left. + a \sin^2 \eta)^3 d\eta \right) \left( \int_0^{\xi_0} \cosh 2\xi \cos^3 \frac{\pi \xi}{2\xi_0} d\xi \right) \\
& \quad + \frac{h^2}{2} \left( \int_0^{2\pi} 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos(2K-1)\eta (1 \right. \\
& \quad \left. + a \sin^2 \eta)^3 d\eta \right) \left( \int_0^{\xi_0} \cosh 2\xi \cos^3 \frac{\pi \xi}{2\xi_0} d\xi \right) \\
& \quad - \frac{h^2}{2} \left( \int_0^{2\pi} J_0(hu) \cos 2\eta (1 + a \sin^2 \eta)^3 d\eta \right) \left( \int_0^{\xi_0} \cos^3 \frac{\pi \xi}{2\xi_0} d\xi \right) \\
& \quad - \frac{h^2}{2} \left( \int_0^{2\pi} 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(hu) \cos 2K\eta \cos 2\eta (1 \right. \\
& \quad \left. + a \sin^2 \eta)^3 d\eta \right) \left( \int_0^{\xi_0} \cos^3 \frac{\pi \xi}{2\xi_0} d\xi \right)
\end{aligned}$$

$$\begin{aligned}
 & - \frac{h^2}{2} \left( \int_0^{2\pi} 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos(2K-1)\eta \cos 2\eta (1 \right. \\
 & \left. + a \sin^2 \eta)^3 d\eta \right) \left( \int_0^{\xi_0} \cos^3 \frac{\pi \xi}{2\xi_0} d\xi \right).
 \end{aligned}$$

Here,

$$\begin{aligned}
 \int_0^{\xi_0} \cosh 2\xi \cos^3 \frac{\pi \xi}{2\xi_0} d\xi &= \frac{3}{2} \xi_0 \pi \cosh 2\xi_0 \left[ \frac{1}{16 \xi_0^2 + \pi^2} \right. \\
 &\quad \left. - \frac{1}{16 \xi_0^2 + 9 \pi^2} \right]
 \end{aligned}$$

$$\int_0^{2\pi} J_0(hu) (1 + a \sin^2 \eta)^3 d\eta = \left( 1 + \frac{3}{2} a + \frac{9}{8} a^2 + \frac{5}{16} a^3 \right) 2\pi J_0(hu)$$

$$\int_0^{2\pi} 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(hu) \cos 2K\eta (1 + a \sin^2 \eta)^3 d\eta =$$

$$\left( \frac{3}{2} a + \frac{3}{2} a^2 + \frac{15}{32} a^3 \right) 2\pi J_2(hu) + \frac{3}{4} a^2 \left( 1 + \frac{a}{2} \right) \pi J_4(hu)$$

$$+ \frac{1}{16} a^3 \pi J_6(hu)$$

$$\int_0^{2\pi} 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos(2K-1)\eta (1 + a \sin^2 \eta)^3 d\eta = 0$$

$$\int_0^{\xi_0} \cos^3 \frac{\pi \xi}{2\xi_0} d\xi = \frac{4\xi_0}{3\pi}$$

$$\int_0^{2\pi} J_0(hu) \cos 2\eta (1 + a \sin^2 \eta)^3 d\eta = - \left( \frac{3}{2}a + \frac{3}{2}a^2 + \frac{15}{32}a^3 \right) \pi J_0(hu)$$

$$\begin{aligned} \int_0^{2\pi} 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(hu) \cos 2K\eta \cos 2\eta (1 + a \sin^2 \eta)^3 d\eta = \\ - \left( 2 + 3a + \frac{21}{8}a^2 + \frac{13}{16}a^3 \right) \pi J_2(hu) - a \left( \frac{3}{2} + \frac{3}{2}a + \frac{1}{2}a^2 \right) \pi J_4(hu) \\ - \frac{3}{8}a^2 \left( 1 + \frac{a}{2} \right) \pi J_6(hu) - \frac{1}{32}a^3 \pi J_8(hu) \end{aligned}$$

$$\int_0^{2\pi} 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos (2K-1)\eta \cos 2\eta (1 + a \sin^2 \eta)^3 d\eta = 0.$$

Therefore,

$$G_3(u, 0) = h^2 \xi_0 \sum_{r=0}^4 \alpha_{2r} J_{2r}(hu)$$

where

$$\begin{aligned} \alpha_0 &= \frac{3}{2} \pi^2 \cosh 2\xi_0 \left[ \frac{1}{16 \xi_0^2 + \pi^2} - \frac{1}{16 \xi_0^2 + 9\pi^2} \right] \left( 1 \right. \\ &\quad \left. + \frac{3}{2}a + \frac{9}{8}a^2 + \frac{5}{16}a^3 \right) + \frac{2}{3}a \left( \frac{3}{2} + \frac{3}{2}a + \frac{15}{32}a^2 \right) \end{aligned}$$

$$\begin{aligned}
\alpha_2 &= \frac{3}{2} \pi^2 \cosh 2\xi_0 \left[ \frac{1}{16 \xi_0^2 + \pi^2} - \frac{1}{16 \xi_0^2 + 9\pi^2} \right] \left( \frac{3}{2} a + \frac{3}{2} a^2 \right. \\
&\quad \left. + \frac{15}{32} a^3 \right) + \frac{2}{3} \left( 2 + 3a + \frac{21}{8} a^2 + \frac{13}{16} a^3 \right) \\
\alpha_4 &= \frac{9}{16} \pi^2 \cosh 2\xi_0 \left[ \frac{1}{16 \xi_0^2 + \pi^2} - \frac{1}{16 \xi_0^2 + 9\pi^2} \right] a^2 \left( 1 + \frac{a}{2} \right) \\
&\quad + \frac{2}{3} a \left( \frac{3}{2} + \frac{3}{2} a + \frac{1}{2} a^2 \right) \\
\alpha_6 &= \frac{3}{64} \pi^2 \cosh 2\xi_0 \left[ \frac{1}{16 \xi_0^2 + \pi^2} - \frac{1}{16 \xi_0^2 + 9\pi^2} \right] a^3 \\
&\quad + \frac{1}{4} a^2 \left( 1 + \frac{a}{2} \right) \\
\alpha_8 &= \frac{1}{48} a^3.
\end{aligned}$$

For N = 4,

$$\begin{aligned}
G_4(u, 0) &= \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} \left[ J_0(hu) + 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(hu) \cos 2K\eta \right. \\
&\quad \left. + 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos(2K-1)\eta \right] \cdot \cosh 2\xi \left[ (1 \right. \\
&\quad \left. + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0} \right]^4 d\xi d\eta
\end{aligned}$$

$$\begin{aligned}
& - \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} \left[ J_0(hu) + 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(hu) \cos 2K\eta \right. \\
& \quad \left. + 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos(2K-1)\eta \right] \cos 2\eta \left[ (1 \right. \\
& \quad \left. + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0} \right]^4 d\xi d\eta \\
& = \frac{h^2}{2} \left( \int_0^{2\pi} J_0(hu) (1 + a \sin^2 \eta)^4 d\eta \right) \left( \int_0^{\xi_0} \cosh 2\xi \cos^4 \frac{\pi \xi}{2\xi_0} d\xi \right) \\
& \quad + \frac{h^2}{2} \left( \int_0^{2\pi} 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(hu) \cos 2K\eta (1 \right. \\
& \quad \left. + a \sin^2 \eta)^4 d\eta \right) \left( \int_0^{\xi_0} \cosh 2\xi_0 \cos^4 \frac{\pi \xi}{2\xi_0} d\xi \right) \\
& \quad + \frac{h^2}{2} \left( \int_0^{2\pi} 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos(2K-1)\eta (1 \right. \\
& \quad \left. + a \sin^2 \eta)^4 d\eta \right) \left( \int_0^{\xi_0} \cosh 2\xi \cos^4 \frac{\pi \xi}{2\xi_0} d\xi \right) \\
& - \frac{h^2}{2} \left( \int_0^{2\pi} J_0(hu) \cos 2\eta (1 + a \sin^2 \eta)^4 d\eta \right) \left( \int_0^{\xi_0} \cos^4 \frac{\pi \xi}{2\xi_0} d\xi \right) \\
& - \frac{h^2}{2} \left( \int_0^{2\pi} 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(hu) \cos 2K\eta \cos 2\eta (1 \right. \\
& \quad \left. + a \sin^2 \eta)^4 d\eta \right) \left( \int_0^{\xi_0} \cos^4 \frac{\pi \xi}{2\xi_0} d\xi \right)
\end{aligned}$$

$$\begin{aligned}
& - \frac{h^2}{2} \left( \int_0^{2\pi} 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos(2K-1)\eta \cos 2\eta (1 \right. \\
& \left. + a \sin^2 \eta)^4 d\eta \right) \left( \int_0^{\xi_0} \cos^4 \frac{\pi \xi}{2\xi_0} d\xi \right).
\end{aligned}$$

Here,

$$\int_0^{\xi_0} \cosh 2\xi \cos^4 \frac{\pi \xi}{2\xi_0} d\xi = \left( \frac{3}{16} - \frac{\xi_0^2}{4 \xi_0^2 + \pi^2} + \frac{\xi_0^2}{16 (\xi_0^2 + \pi^2)} \right) \sinh 2\xi_0$$

$$\int_0^{2\pi} J_0(hu) (1 + a \sin^2 \eta)^4 d\eta = 2\pi \left( 1 + 2a + \frac{9}{4} a^2 + \frac{5}{4} a^3 + \frac{35}{128} a^4 \right) J_0(hu)$$

$$\begin{aligned}
& \int_0^{2\pi} 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(hu) \cos 2K\eta (1 + a \sin^2 \eta)^4 d\eta = 2\pi \left( 2a \right. \\
& \left. + 3a^2 + \frac{15}{8} a^3 + \frac{7}{16} a^4 \right) J_2(hu) + 2\pi \left( \frac{3}{4} a^2 + \frac{3}{4} a^3 + \frac{7}{32} a^4 \right) J_4(hu) \\
& + 2\pi \left( \frac{1}{8} a^3 + \frac{1}{16} a^4 \right) J_6(hu) + 2\pi \frac{1}{128} a^4 J_8(hu)
\end{aligned}$$

$$\int_0^{2\pi} 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos(2K-1)\eta (1 + a \sin^2 \eta)^4 d\eta = 0$$

$$\int_0^{\xi_0} \cos^4 \frac{\pi \xi}{2\xi_0} d\xi = \frac{3}{8} \xi_0$$

$$\int_0^{2\pi} J_0(hu) \cos 2\eta (1 + a \sin^2 \eta)^4 d\eta = - \left( 2a + 3a^2 + \frac{15}{8} a^3 + \frac{7}{16} a^4 \right) \pi J_0(hu)$$

$$\begin{aligned} \int_0^{2\pi} 2 \sum_{K=1}^{\infty} (-)^K J_{2K}(hu) \cos 2K \eta \cos 2\eta (1 + a \sin^2 \eta)^4 d\eta &= - \left( 2 + 4a + \frac{21}{4} a^2 + \frac{13}{4} a^3 + \frac{49}{64} a^4 \right) \pi J_2(hu) - \left( 2a + 3a^2 + 2a^3 + \frac{1}{2} a^4 \right) \pi J_4(hu) - \left( \frac{3}{4} a^2 + \frac{3}{4} a^3 + \frac{29}{128} a^4 \right) \pi J_6(hu) \\ &\quad - \left( \frac{1}{8} a^3 + \frac{1}{16} a^4 \right) \pi J_8(hu) - \frac{1}{128} a^4 \pi J_{10}(hu) \\ \int_0^{2\pi} 2i \sum_{K=1}^{\infty} (-)^{K-1} J_{2K-1}(hu) \cos (2K-1) \eta \cos 2\eta (1 + a \sin^2 \eta)^4 d\eta &= 0. \end{aligned}$$

Therefore,

$$G_4(u, 0) = h^2 \pi \sum_{r=0}^5 \beta_{2r} J_{2r}(hu)$$

where

$$\begin{aligned} \beta_0 &= \left( 1 + 2a + \frac{9}{4} a^2 + \frac{5}{4} a^3 + \frac{35}{128} a^4 \right) \left( \frac{3}{16} - \frac{\xi_0^2}{4\xi_0^2 + \pi^2} \right. \\ &\quad \left. + \frac{\xi_0^2}{16(\xi_0^2 + \pi^2)} \right) \sinh 2\xi_0 + \frac{3}{16} \xi_0 \left( 2a + 3a^2 + \frac{15}{8} a^3 + \frac{7}{16} a^4 \right) \end{aligned}$$

$$\begin{aligned}
\beta_2 &= \left(2a + 3a^2 + \frac{15}{8}a^3 + \frac{7}{16}a^4\right) \left(\frac{3}{16} - \frac{\xi_0^2}{4\xi_0^2 + \pi^2}\right. \\
&\quad \left. + \frac{\xi_0^2}{16(\xi_0^2 + \pi^2)}\right) \sinh 2\xi_0 + \frac{3}{16}\xi_0 \left(2 + 4a + \frac{21}{4}a^2\right. \\
&\quad \left. + \frac{13}{4}a^3 + \frac{49}{64}a^4\right) \\
\beta_4 &= \left(\frac{3}{4}a^2 + \frac{3}{4}a^3 + \frac{7}{32}a^4\right) \left(\frac{3}{16} - \frac{\xi_0^2}{4\xi_0^2 + \pi^2} + \frac{\xi_0^2}{16(\xi_0^2 + \pi^2)}\right) \sinh 2\xi_0 \\
&\quad + \frac{3}{16}\xi_0 \left(2a + 3a^2 + 2a^3 + \frac{1}{2}a^4\right) \\
\beta_6 &= \frac{1}{8}a^3 \left(1 + \frac{1}{2}a\right) \left(\frac{3}{16} - \frac{\xi_0^2}{4\xi_0^2 + \pi^2} + \frac{\xi_0^2}{16(\xi_0^2 + \pi^2)}\right) \sinh 2\xi_0 \\
&\quad + \frac{3}{16}\xi_0 \left(\frac{3}{4}a^2 + \frac{3}{4}a^3 + \frac{29}{128}a^4\right) \\
\beta_8 &= \frac{1}{128}a^4 \left(\frac{3}{16} - \frac{\xi_0^2}{4\xi_0^2 + \pi^2} + \frac{\xi_0^2}{16(\xi_0^2 + \pi^2)}\right) \sinh 2\xi_0 \\
&\quad + \frac{3}{128}\xi_0 a^3 \left(1 + \frac{1}{2}a\right) \\
\beta_{10} &= \frac{3}{2048}\xi_0 a^4.
\end{aligned}$$

For the zeroth moment in elliptical coordinates,

$$\begin{aligned}
 \mu_0 &= \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} (\cosh 2\xi - \cos 2\eta) F^2(\xi, \eta) d\xi d\eta \\
 &= \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} (\cosh 2\xi - \cos 2\eta) \left[ (1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0} \right]^{2N} d\xi d\eta \\
 &= \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} \cosh 2\xi \left[ (1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0} \right]^{2N} d\xi d\eta \\
 &- \frac{h^2}{2} \int_0^{2\pi} \int_0^{\xi_0} \cos 2\eta \left[ (1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0} \right]^{2N} d\xi d\eta.
 \end{aligned}$$

For N = 1,

$$\begin{aligned}
 \mu_{0,1} &= \frac{h^2}{2} \left( \int_0^{2\pi} (1 + a \sin^2 \eta)^2 d\eta \right) \left( \int_0^{\xi_0} \cosh 2\xi \cos^2 \frac{\pi \xi}{2\xi_0} d\xi \right) \\
 &- \frac{h^2}{2} \left( \int_0^{2\pi} \cos 2\eta (1 + a \sin^2 \eta)^2 d\eta \right) \left( \int_0^{\xi_0} \cos^2 \frac{\pi \xi}{2\xi_0} d\xi \right).
 \end{aligned}$$

Here,

$$\begin{aligned}
 \int_0^{2\pi} (1 + a \sin^2 \eta)^2 d\eta &= 2\pi \left( 1 + a + \frac{3}{8} a^2 \right) \\
 \int_0^{\xi_0} \cosh 2\xi \cos^2 \frac{\pi \xi}{2\xi_0} d\xi &= \frac{\pi^2 \sinh 2\xi_0}{4(4\xi_0^2 + \pi^2)}
 \end{aligned}$$

$$\int_0^{2\pi} \cos 2\eta (1 + a \sin^2 \eta)^2 d\eta = -a \left(1 + \frac{a}{2}\right) \pi$$

$$\int_0^{\xi_0} \cos^2 \frac{\pi \xi}{2\xi_0} d\xi = \frac{1}{2} \xi_0 .$$

Therefore,

$$\mu_{0,1} = h^2 \pi \left[ \left(1 + a + \frac{3}{8} a^2\right) \frac{\pi^2 \sinh 2\xi_0}{4(4\xi_0^2 + \pi^2)} + \frac{1}{4} \xi_0 a \left(1 + \frac{1}{2} a\right) \right] . \quad (1)$$

For N = 2,

$$\begin{aligned} \mu_{0,2} &= \frac{h^2}{2} \left( \int_0^{2\pi} (1 + a \sin^2 \eta)^4 d\eta \right) \left( \int_0^{\xi_0} \cosh 2\xi \cos^4 \frac{\pi \xi}{2\xi_0} d\xi \right) \\ &\quad - \frac{h^2}{2} \left( \int_0^{2\pi} \cos 2\eta (1 + a \sin^2 \eta)^4 d\eta \right) \left( \int_0^{\xi_0} \cos^4 \left(\frac{\pi \xi}{2\xi_0}\right) d\xi \right) \end{aligned}$$

Here,

$$\int_0^{2\pi} (1 + a \sin^2 \eta)^4 d\eta = \left(1 + 2a + \frac{9}{4} a^2 + \frac{5}{4} a^3 + \frac{35}{128} a^4\right) 2\pi$$

$$\int_0^{\xi_0} \cosh 2\xi \cos^4 \frac{\pi \xi}{2\xi_0} d\xi = \left( \frac{3}{16} - \frac{\xi_0^2}{4\xi_0^2 + \pi^2} + \frac{\xi_0^2}{16(\xi_0^2 + \pi^2)} \right) \sinh 2\xi_0$$

$$\int_0^{2\pi} \cos 2\eta (1 + a \sin^2 \eta)^4 d\eta = - \left(2a + 3a^2 + \frac{15}{8} a^3 + \frac{7}{16} a^4\right) \pi$$

$$\int_0^{\xi_0} \cos^4 \frac{\pi \xi}{2\xi_0} d\xi = \frac{3}{8} \xi_0.$$

Therefore,

$$\begin{aligned} u_{0,2} &= h^2 \pi \left[ \left( 1 + 2a + \frac{9}{4} a^2 + \frac{5}{4} a^3 + \frac{35}{128} a^4 \right) \left( \frac{3}{16} \right. \right. \\ &\quad \left. \left. - \frac{\xi_0^2}{4 \xi_0^2 + \pi^2} + \frac{\xi_0^2}{16 (\xi_0^2 + \pi^2)} \right) \sinh 2 \xi_0 \right. \\ &\quad \left. + \frac{3}{16} \xi_0 \left( 2a + 3a^2 + \frac{15}{8} a^3 + \frac{7}{16} a^4 \right) \right]. \end{aligned} \quad (2)$$

For N = 3,

$$\begin{aligned} u_{0,3} &= \frac{h^2}{2} \left( \int_0^{2\pi} (1 + a \sin^2 \eta)^6 d\eta \right) \left( \int_0^{\xi_0} \cosh 2\xi \cos^6 \frac{\pi \xi}{2\xi_0} d\xi \right) \\ &\quad - \frac{h^2}{2} \left( \int_0^{2\pi} \cos 2\eta (1 + a \sin^2 \eta)^6 d\eta \right) \left( \int_0^{\xi_0} \cos^6 \frac{\pi \xi}{2\xi_0} d\xi \right). \end{aligned}$$

Here,

$$\begin{aligned} \int_0^{2\pi} (1 + a \sin^2 \eta)^6 d\eta &= 2\pi \left( 1 + 3a + \frac{45}{8} a^2 + \frac{25}{4} a^3 + \frac{525}{128} a^4 \right. \\ &\quad \left. + \frac{189}{128} a^5 + \frac{231}{1024} a^6 \right) \\ \int_0^{\xi_0} \cosh 2\xi \cos^6 \frac{\pi \xi}{2\xi_0} d\xi &= \left( \frac{5}{32} - \frac{15 \xi_0^2}{16 (4 \xi_0^2 + \pi^2)} + \frac{3 \xi_0^2}{32 (\xi_0^2 + \pi^2)} \right. \\ &\quad \left. - \frac{\xi_0^2}{16 (\xi_0^2 + 9 \pi^2)} \right) \sinh 2 \xi_0 \end{aligned}$$

$$\int_0^{2\pi} \cos 2\eta (1 + a \sin^2 \eta)^6 d\eta = -\pi a \left( 3 + \frac{15}{2} a + \frac{75}{8} a^2 + \frac{105}{16} a^3 + \frac{315}{128} a^4 + \frac{99}{256} a^5 \right)$$

$$\int_0^{\xi_0} \cos^6 \frac{\pi \xi}{2\xi_0} d\xi = \frac{5}{16} \xi_0.$$

Therefore,

$$\mu_{0,3} = \frac{1}{8} h^2 \pi \left[ \left( 1 + 3a + \frac{45}{8} a^2 + \frac{25}{4} a^3 + \frac{525}{128} a^4 + \frac{189}{128} a^5 + \frac{231}{1024} a^6 \right) \left( \frac{5}{4} - \frac{15 \xi_0^2}{2(4\xi_0^2 + \pi^2)} + \frac{3 \xi_0^2}{4(\xi_0^2 + \pi^2)} - \frac{\xi_0^2}{2(4\xi_0^2 + 9\pi^2)} \right) \sinh 2\xi_0 + \frac{5}{4} \xi_0 a \left( 3 + \frac{15}{2} a + \frac{75}{8} a^2 + \frac{105}{16} a^3 + \frac{315}{128} a^4 + \frac{99}{256} a^5 \right) \right]. \quad (3)$$

For N = 4,

$$\mu_{0,4} = \frac{h^2}{2} \left( \int_0^{2\pi} (1 + a \sin^2 \eta)^8 d\eta \right) \left( \int_0^{\xi_0} \cosh 2\xi \cos^8 \frac{\pi \xi}{2\xi_0} d\xi \right)$$

$$- \frac{h^2}{2} \left( \int_0^{2\pi} \cos 2\eta (1 + a \sin^2 \eta)^8 d\eta \right) \left( \int_0^{\xi_0} \cos^8 \frac{\pi \xi}{2\xi_0} d\xi \right).$$

Here,

$$\int_0^{2\pi} (1 + a \sin^2 \eta)^8 d\eta = 2\pi \left( 1 + 4a + \frac{21}{2} a^2 + \frac{35}{2} a^3 + \frac{1225}{64} a^4 + \frac{441}{32} a^5 + \frac{1617}{128} a^6 + \frac{429}{128} a^7 + \frac{6335}{32768} a^8 \right)$$

$$\int_0^{\xi_0} \cosh 2\xi \cos^8 \frac{\pi\xi}{2\xi_0} d\xi = \left( \frac{35}{256} - \frac{7\xi_0^2}{32(4\xi_0^2 + \pi^2)} + \frac{7\xi_0^2}{64(\xi_0^2 + \pi^2)} - \frac{\xi_0^2}{8(4\xi_0^2 + 9\pi^2)} + \frac{\xi_0^2}{256(\xi_0^2 + 4\pi^2)} \right) \sinh 2\xi_0$$

$$\int_0^{2\pi} \cos 2\eta (1 + a \sin^2 \eta)^8 d\eta = -\pi a \left( 1 + \frac{a}{2} \right) \left( 4 + 12a + \frac{81}{4} a^2 + \frac{41}{2} a^3 + \frac{407}{32} a^4 + \frac{143}{32} a^5 + \frac{715}{1024} a^6 \right)$$

$$\int_0^{\xi_0} \cos^8 \frac{\pi\xi}{2\xi_0} d\xi = \frac{35}{128} \xi_0.$$

Therefore,

$$\begin{aligned} \mu_{0,4} &= \frac{1}{4} h^2 \pi \left[ \left( 1 + 4a + \frac{21}{2} a^2 + \frac{35}{2} a^3 + \frac{1225}{64} a^4 + \frac{441}{32} a^5 + \frac{1617}{128} a^6 + \frac{429}{128} a^7 + \frac{6335}{32768} a^8 \right) \left( \frac{35}{64} - \frac{7\xi_0^2}{8(4\xi_0^2 + \pi^2)} + \frac{7\xi_0^2}{16(\xi_0^2 + \pi^2)} - \frac{\xi_0^2}{2(4\xi_0^2 + \pi^2)} + \frac{\xi_0^2}{64(\xi_0^2 + 4\pi^2)} \right) \sinh 2\xi_0 + \frac{35}{64} \xi_0 a \left( 1 + \frac{a}{2} \right) \right] \end{aligned}$$

$$\cdot \left( 4 + 12a + \frac{81}{4} a^2 + \frac{41}{2} a^3 + \frac{407}{32} a^4 + \frac{143}{32} a^5 + \frac{715}{1024} a^6 \right) \Big]. \quad (4)$$

For the second moment in elliptical coordinates,

$$\begin{aligned} \mu_2 &= \int_0^{2\pi} \int_0^{\xi_0} \left[ \left( \frac{\partial F}{\partial \xi} \right)^2 + \left( \frac{\partial F}{\partial \eta} \right)^2 \right] d\xi d\eta \\ &= \int_0^{2\pi} \int_0^{\xi_0} \left[ \frac{N^2 \pi^2}{4 \xi_0^2} (1 + a \sin^2 \eta)^{2N} \cos^{2(N-1)} \frac{\pi \xi}{2\xi_0} \sin^2 \frac{\pi \xi}{2\xi_0} \right] d\xi d\eta \\ &\quad + \int_0^{2\pi} \int_0^{\xi_0} \left[ a^2 N^2 (1 + a \sin^2 \eta)^{2(N-1)} \sin^2 2\eta \cos^{2N} \frac{\pi \xi}{2\xi_0} \right] d\xi d\eta. \end{aligned}$$

For  $N = 1$ ,

$$\begin{aligned} \mu_{2,1} &= \frac{\pi^2}{4 \xi_0^2} \left( \int_0^{2\pi} (1 + a \sin^2 \eta) d\eta \right) \left( \int_0^{\xi_0} \sin^2 \frac{\pi \xi}{2\xi_0} d\xi \right) \\ &\quad + a^2 \left( \int_0^{2\pi} \sin^2 2\eta d\eta \right) \left( \int_0^{\xi_0} \cos^2 \frac{\pi \xi}{2\xi_0} d\xi \right). \end{aligned}$$

Here,

$$\begin{aligned} \int_0^{2\pi} (1 + a \sin^2 \eta)^2 d\eta &= 2\pi \left( 1 + a + \frac{3}{8} a^2 \right) \\ \int_0^{\xi_0} \sin^2 \frac{\pi \xi}{2\xi_0} d\xi &= \frac{1}{2} \xi_0 \end{aligned}$$

$$\int_0^{2\pi} \sin^2 2\eta \ d\eta = \pi$$

$$\int_0^{\xi_0} \cos^2 \frac{\pi\xi}{2\xi_0} \ d\xi = \frac{1}{2} \xi_0 .$$

Therefore,

$$\mu_{2,1} = \frac{\pi^3}{4\xi_0^2} \left( 1 + a + \frac{3}{8} a^2 \right) + \frac{\pi}{2} a^2 \xi_0 . \quad (5)$$

For N = 2,

$$\begin{aligned} \mu_{2,2} &= \frac{\pi^2}{\xi_0^2} \left( \int_0^{2\pi} (1 + a \sin^2 \eta)^4 \ d\eta \right) \left( \int_0^{\xi_0} \frac{1}{4} \sin^2 \frac{\pi\xi}{\xi_0} \ d\xi \right) \\ &\quad + 4a^2 \left( \int_0^{2\pi} (1 + a \sin^2 \eta)^2 \sin^2 2\eta \ d\eta \right) \left( \int_0^{\xi_0} \cos^4 \frac{\pi\xi}{2\xi_0} \ d\xi \right). \end{aligned}$$

Here,

$$\int_0^{2\pi} (1 + a \sin^2 \eta)^4 \ d\eta = 2\pi \left( 1 + 2a + \frac{9}{4} a^2 + \frac{5}{4} a^3 + \frac{35}{128} a^4 \right)$$

$$\int_0^{\xi_0} \frac{1}{4} \sin^2 \frac{\pi\xi}{\xi_0} \ d\xi = \frac{1}{8} \xi_0$$

$$\int_0^{2\pi} (1 + a \sin^2 \eta)^2 \sin^2 2\eta \ d\eta = \pi \left( 1 + a + \frac{5}{16} a^2 \right)$$

$$\int_0^{\xi_0} \cos^4 \frac{\pi \xi}{2\xi_0} \quad d\xi = \frac{3}{8} \xi_0 .$$

Therefore,

$$\begin{aligned} \mu_{2,2} &= \frac{\pi^3}{4\xi_0^2} \left( 1 + 2a + \frac{9}{4} a^2 + \frac{5}{4} a^3 + \frac{35}{128} a^4 \right) + \frac{3}{2} \pi \xi_0 \left( 1 + a \right. \\ &\quad \left. + \frac{5}{16} a^2 \right) a^2 . \end{aligned} \quad (6)$$

For N = 3,

$$\begin{aligned} \mu_{2,3} &= \frac{9\pi^2}{4\xi_0^2} \left( \int_0^{2\pi} (1 + a \sin^2 \eta)^6 d\eta \right) \left( \int_0^{\xi_0} \cos^4 \frac{\pi \xi}{2\xi_0} \sin^2 \frac{\pi \xi}{2\xi_0} d\xi \right) \\ &\quad + 9a^2 \left( \int_0^{2\pi} (1 + a \sin^2 \eta)^4 \sin^2 2\eta d\eta \right) \left( \int_0^{\xi_0} \cos^6 \frac{\pi \xi}{2\xi_0} d\xi \right) . \end{aligned}$$

Here,

$$\begin{aligned} \int_0^{2\pi} (1 + a \sin^2 \eta)^6 d\eta &= 2\pi \left( 1 + 3a + \frac{45}{8} a^2 + \frac{25}{4} a^3 + \frac{525}{128} a^4 \right. \\ &\quad \left. + \frac{189}{128} a^5 + \frac{231}{1024} a^6 \right) \end{aligned}$$

$$\int_0^{\xi_0} \cos^4 \frac{\pi \xi}{2\xi_0} \sin^2 \frac{\pi \xi}{2\xi_0} d\xi = \frac{1}{16} \xi_0$$

$$\int_0^{2\pi} (1 + a \sin^2 \eta)^4 \sin^2 2\eta d\eta = \pi \left( 1 + 2a + \frac{15}{8} a^2 + \frac{7}{8} a^3 + \frac{21}{32} a^4 \right)$$

$$\int_0^{\xi_0} \cos^6 \frac{\pi \xi}{2\xi_0} d\xi = \frac{5}{16} \xi_0.$$

Therefore,

$$\begin{aligned} \mu_{2,3} &= \frac{9\pi^3}{32\xi_0^2} \left( 1 + 3a + \frac{45}{8}a^2 + \frac{25}{4}a^3 + \frac{525}{128}a^4 + \frac{189}{128}a^5 \right. \\ &\quad \left. + \frac{231}{1024}a^6 \right) + \frac{45}{16}\xi_0 a^2 \pi \left( 1 + 2a + \frac{15}{8}a^2 + \frac{7}{8}a^3 + \frac{21}{32}a^4 \right). \end{aligned}$$

For N = 4,

$$\begin{aligned} \mu_{2,4} &= \frac{4\pi^2}{\xi_0^2} \left( \int_0^{2\pi} (1 + a \sin^2 \eta)^8 d\eta \right) \left( \int_0^{\xi_0} \cos^6 \frac{\pi \xi}{2\xi_0} \sin^2 \frac{\pi \xi}{2\xi_0} d\xi \right) \\ &\quad + 16a^2 \left( \int_0^{2\pi} (1 + a \sin^2 \eta)^6 \sin^2 2\eta d\eta \right) \left( \int_0^{\xi_0} \cos^8 \frac{\pi \xi}{2\xi_0} d\xi \right). \end{aligned}$$

Here,

$$\begin{aligned} \int_0^{2\pi} (1 + a \sin^2 \eta)^8 d\eta &= 2\pi \left( 1 + 4a + \frac{21}{2}a^2 + \frac{35}{2}a^3 + \frac{1225}{64}a^4 \right. \\ &\quad \left. + \frac{441}{32}a^5 + \frac{1617}{128}a^6 + \frac{429}{128}a^7 + \frac{6335}{32768}a^8 \right) \end{aligned}$$

$$\int_0^{\xi_0} \cos^6 \frac{\pi \xi}{2\xi_0} \sin^2 \frac{\pi \xi}{2\xi_0} d\xi = \frac{5}{128} \xi_0$$

$$\int_0^{2\pi} (1 + a \sin^2 \eta)^6 \sin^2 2\eta d\eta = \left( 1 + 3a + \frac{75}{16} a^2 + \frac{35}{8} a^3 + \frac{75}{128} a^4 + \frac{99}{128} a^5 + \frac{429}{4096} a^6 \right) \pi$$

$$\int_0^{\xi_0} \cos^8 \frac{\pi \xi}{2\xi_0} d\xi = \frac{55}{16} \xi_0.$$

Therefore,

$$\begin{aligned} \mu_{2,4} &= \frac{5}{16} \frac{\pi^3}{\xi_0} \left( 1 + 4a + \frac{21}{2} a^2 + \frac{35}{2} a^3 + \frac{1225}{64} a^4 + \frac{441}{32} a^5 \right. \\ &\quad \left. + \frac{1617}{128} a^6 + \frac{429}{128} a^7 + \frac{6335}{32768} a^8 \right) + 55 \xi_0 \pi a^2 \left( 1 + 3a + \frac{75}{16} a^2 \right. \\ &\quad \left. + \frac{35}{8} a^3 + \frac{75}{128} a^4 + \frac{99}{128} a^5 + \frac{429}{4096} a^6 \right). \end{aligned} \quad (8)$$

### III. GRAPHS

Figures 1 through 5 are plots of illuminations and far-field power patterns. All illuminations are plotted with peak amplitude equal to unity. For all far-field powers, the logarithm of the power is plotted with the center of the main lobe normalized to zero decibels.

#### FIGURES:

Fig. 1 =  $\left[ (1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2 \xi_0} \right]^N$  Illumination of Elliptical Aperture Along Major Axis.  $N = 1, 2, 3, 4$

Fig. 2 = Far-Field Power for  $\left[ (1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2 \xi_0} \right]$  Elliptical Illumination Major Axis

Fig. 3 = Far-Field Power for  $\left[ (1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2 \xi_0} \right]^2$  Elliptical Illumination Along Major Axis.

Fig. 4 = Far-Field Power for  $\left[ (1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2 \xi_0} \right]^3$  Elliptical Illumination Along Major Axis.

Fig. 5 = Far-Field Power for  $\left[ (1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2 \xi_0} \right]^4$  Elliptical Illumination Along Major Axis.

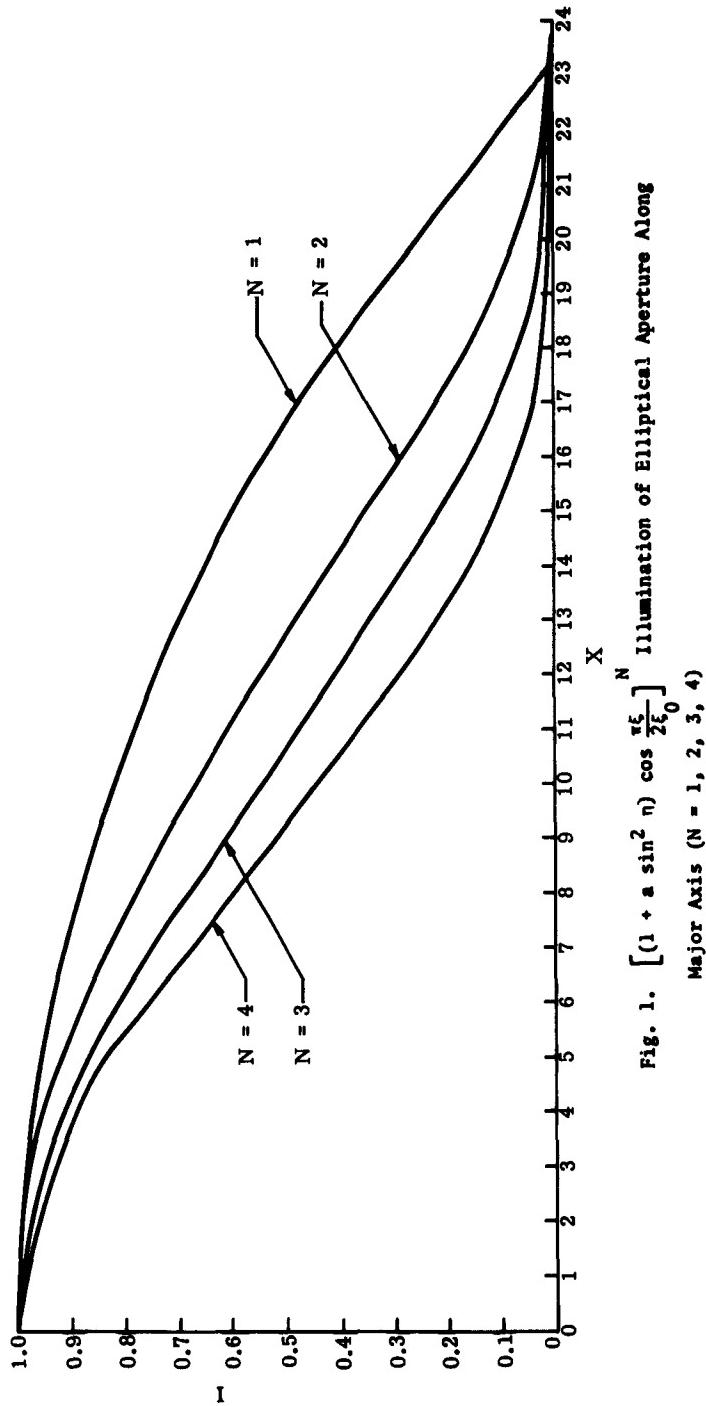


Fig. 1.  $\left[ (1 + \sin^2 n) \cos \frac{n\epsilon}{2\epsilon_0} \right]^N$  Illumination of Elliptical Aperture Along Major Axis ( $N = 1, 2, 3, 4$ )

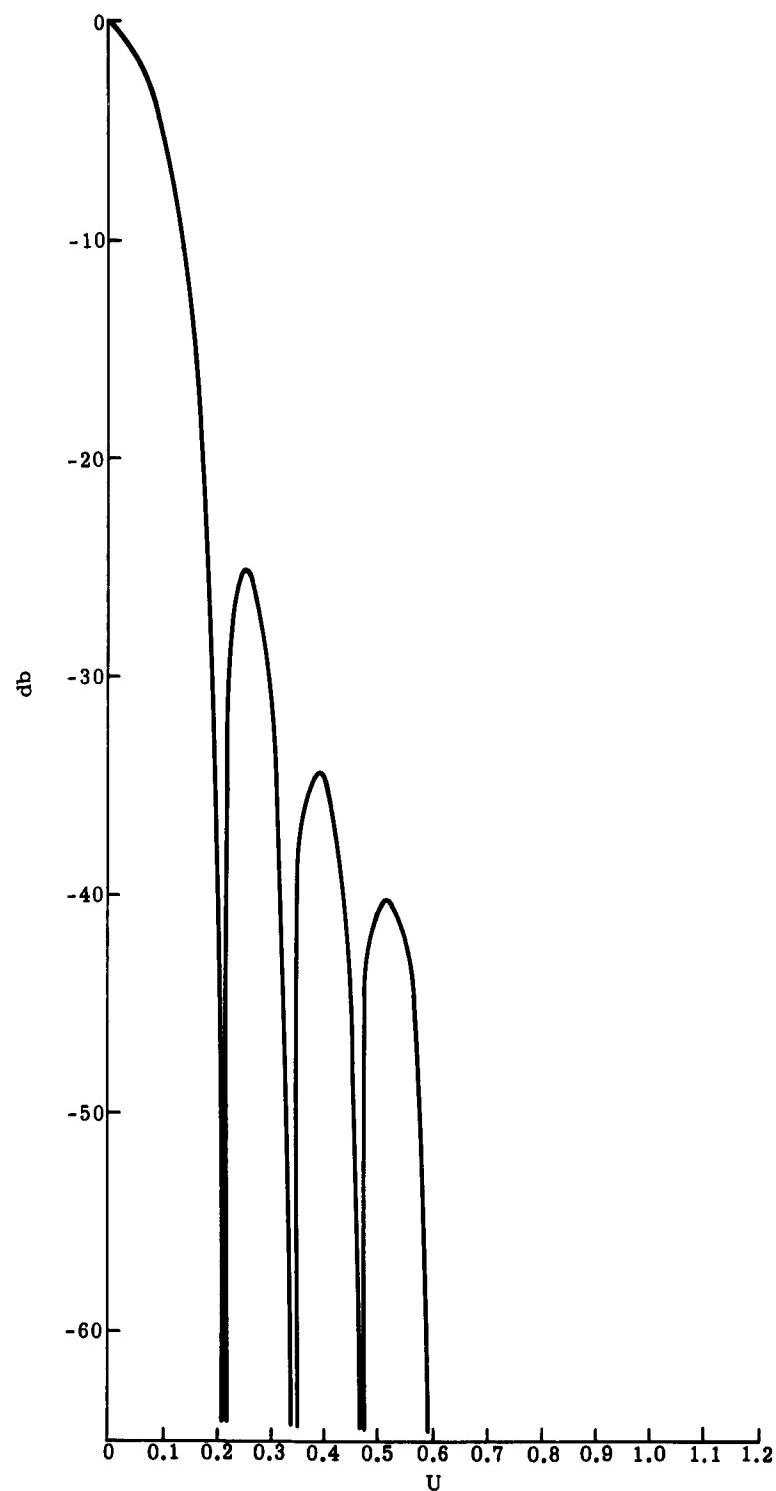


Fig. 2. Far-Field Power for  $\left[ (1 + a \sin^2 \theta) \cos \frac{\pi \xi}{2c_0} \right]$  Elliptical Illumination  
Along Major Axis

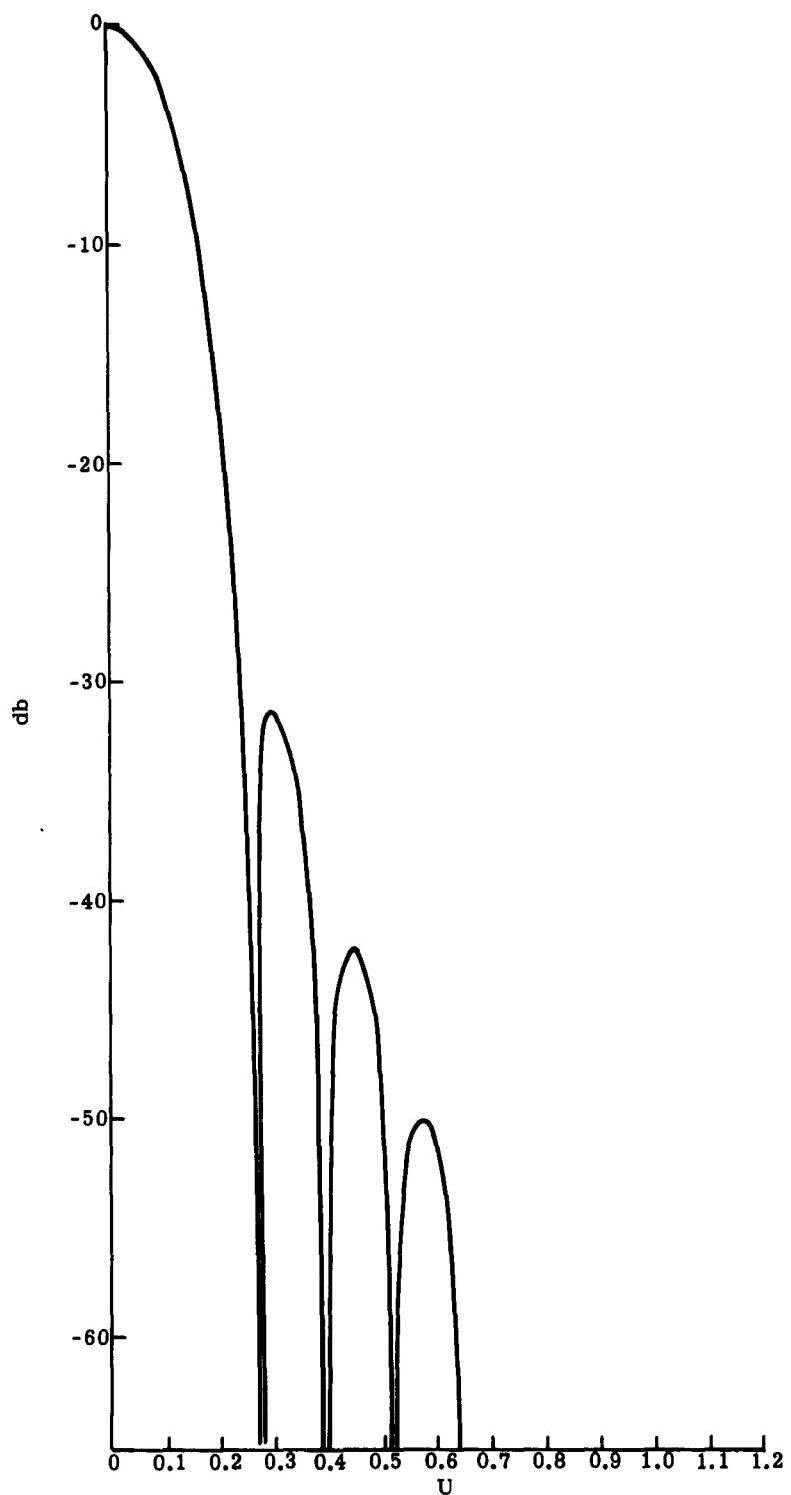


Fig. 3. Far-Field Power for  $\left[ (1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0} \right]^2$  Elliptical Illumination  
Along Major Axis

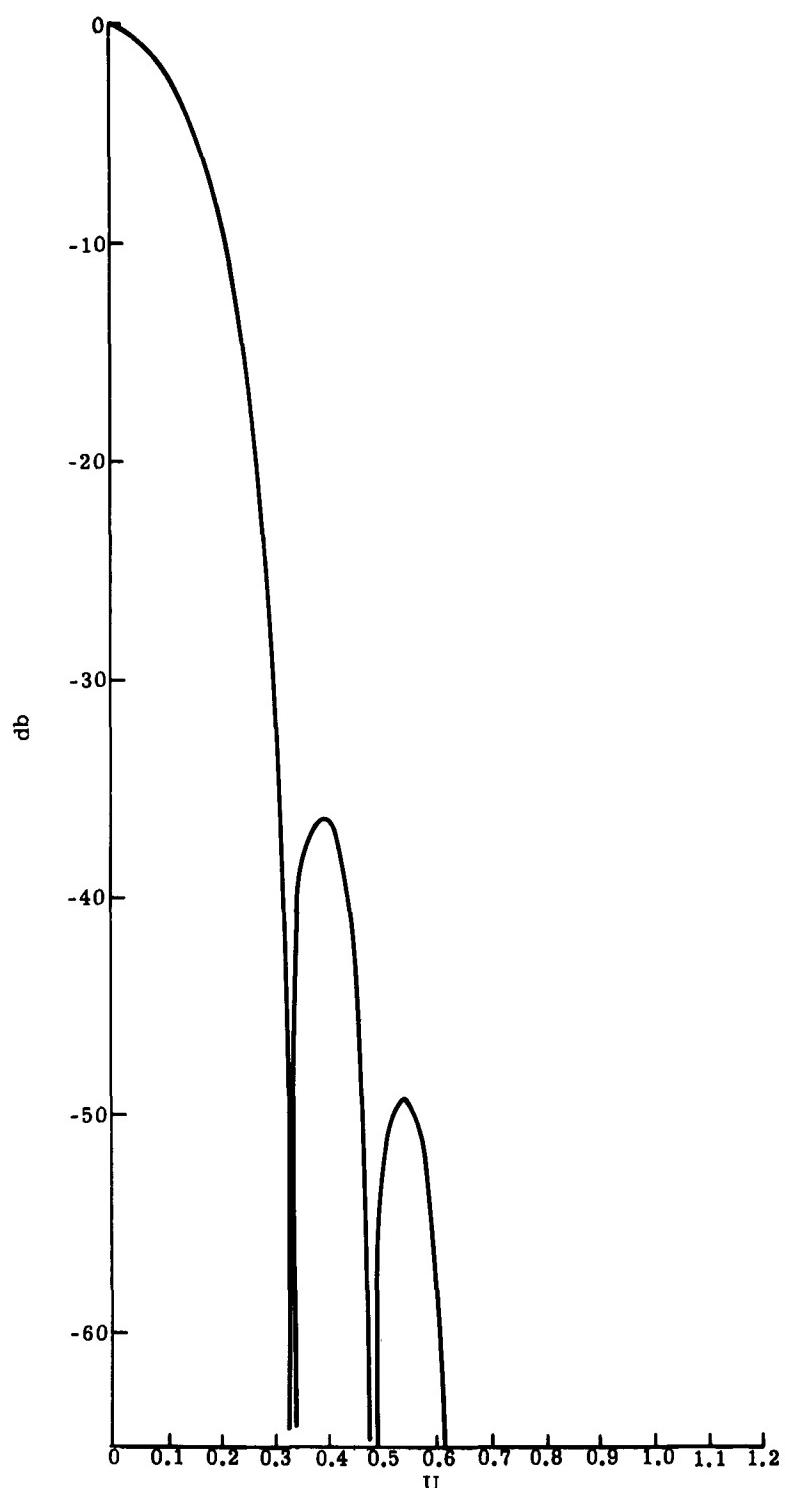


Fig. 4. Far-Field Power for  $\left[ (1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2\epsilon_0} \right]^3$  Elliptical Illumination  
Along Major Axis

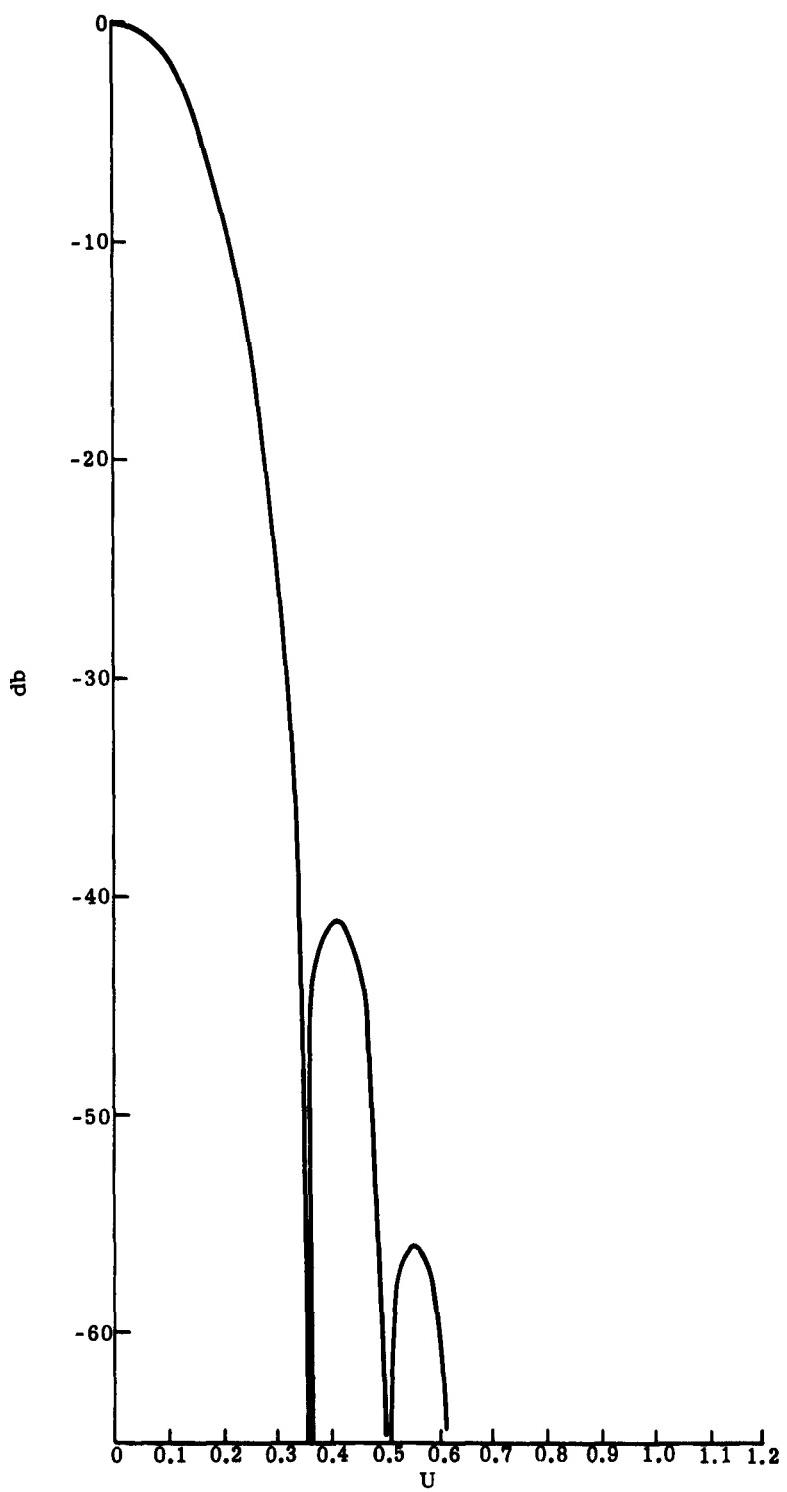


Fig. 5. Far-Field Power for  $\left[ (1 + a \sin^2 n) \cos \frac{\pi \xi}{2\xi_0} \right]^4$  Elliptical Illumination  
Along Major Axis

IV. TABLE OF COMPARISON

Type	Illumination	Function	Beamwidth	Sidelobe	Moment
Circular	Optimum	$J_0(K_{00}r)$	2.2	-28.4	5.794
Circular	Uniform	A constant	1.6	-16.8	$\infty$
Circular	Nonoptimum	$\cos\left(\frac{\pi r}{2a}\right)$	2.1	-25.6	5.83
Circular	Nonoptimum	$\cos^2\left(\frac{\pi r}{2a}\right)$	2.3	-34	7.17
Circular	Nonoptimum	$\cos^3\left(\frac{\pi r}{2a}\right)$	2.6	-41	9.41
Elliptical	Optimum	$ce_0(\xi, q) ce_0(\eta, q)$	0.11	-36	0.0718
Elliptical	Uniform	A constant	0.075	-17.5	$\infty$
Elliptical	Nonoptimum	$(1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0}$	0.08	-24	0.0745
Elliptical	Nonoptimum	$\left[(1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0}\right]^2$	0.095	-30	0.0962
Elliptical	Nonoptimum	$\left[(1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0}\right]^3$	0.115	-36.25	0.1455
Elliptical	Nonoptimum	$\left[(1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0}\right]^4$	0.135	-41	0.2583
Square	Optimum	$\cos \frac{\pi x}{2a} \cos \frac{\pi y}{2a}$	2.0	-22.8	4.9868

**APPENDIX A**

**COMPUTER PROGRAMS**

Program 1.

This program is designed to compute moments of the four selected nonoptimum illuminations for elliptical antennas.

$$\left( \left[ (1 + a \sin^2 \eta) \cos \frac{\pi \xi}{2\xi_0} \right]^N ; N = 1, 2, 3, 4 \right)$$

The value of  $a$  is chosen to be 99.

The program calculates Eqs (1) through (8) and the moments are obtained by

$$(5) \quad \bar{(1)}, \quad (6) \quad \bar{(2)}, \quad (7) \quad \bar{(3)}, \quad (8) \quad \bar{(4)}, \text{ respectively.}$$

```

07300      PRINT2
07324      2 FORMAT(19H TABLE OF MOMENTS/)
07392      AH=23.1
07416      PI=3.14159
07440      SI=0.277
07464      A=99.0
07488      R=(AH**2)*PI*A*SI*(1.+5*A)/4.
07512      S=.1457*PI**2/(4.*SI**2+PI**2)
07536      T=(AH**2)*PI*(1.+A*(1.+0.375*A))
07560      ZU01=T*S*R
07588      PRINT 22,ZU01
07592      22 FORMAT(1IH MU(0,1)=E15.8)
08044      ZU21=(PI**3/(4.*SI))*(1.+A*(1.+0.375*A))+1.5708*(A**2)*SI
08284      PRINT 21,ZU21
08308      21 FORMAT(1IH MU(2,1)=E15.8)
08360      V=ZU21/ZU01
08396      PRINT 1,V
08420      1 FORMAT(5H MOMENT=E15.8///)
08462      P=PI**2
08518      Q=SI**2
08554      PA=(1.-A*(2.+A*(2.25+A*(1.25+(35.*A)/128.)))*
08586      PSI=SI*(.375-SI*.5828*(1./4.*P)-.0625/(P+Q)))
08618      PA1=A*(2.+A*(3.+A*(1.875+A*.4375)))
08646      PA1=SI*.1875*PI
09034      YU02=AH**2*(PI)*(PA*PSI+PA1)
09142      PRINT 31,YU02
09166      31 FORMAT(1IH MU(0,2)=E15.8)
09218      RA=1.5*PI*SI*(1.+A*(1.+0.3125*A))*(A**2)
09358      RIA=(0.25*(PI**3)/SI)+PA+RA
09462      PRINT 32,RIA
09506      32 FORMAT(1IH MU(2,2)=E15.8)
09558      W=RIA/YU02
09594      PRINT 1,W
09618      H=(2.5-.5828*SI*(7.5/(4.*P+.75/(P+Q)+.5/(4.*P+9.*P)))
09918      Q=3.*A*(7.5+A*(9.375*A*(6.5626*A*(2.38125+0.38671*A)))
10062      E=1.+A*(3.+A*(5.625+A*(6.25+A*(4.101+A*(1.476+.2255*A))))))
10230      U03=(AH**2)*PI*SI*(0.125*(E*H+.15625*A*Q))
10410      PRINT 42,U03
10434      42 FORMAT(1IH MU(0,3)=E15.8)
10486      E1=(0.28125*(PI**3)/SI)*E
10558      F=(1.-A*(2.+A*(1.875+A*(.875+0.65625*A))))
10678      F=2.8125*PI*SI*(A**2)*F
10786      U23=E1+F1
10822      PRINT 41,U23
10846      41 FORMAT(1IH MU(2,3)=E15.8)
10898      O=U23/U03
11034      PRINT 1,O
11058      C=10.5
11062      D=17.5
11066      E=19.140625
11030      F=13.78125
11054      G=12.6328125
11078      H=3.3515625
11102      P=6335./32768.
11138      SA=(1.+A*(C+A*(D+A*(E+A*(F+A*(G+A*(H+A*P)))))))
11154      SI=7.0/(4.*SI**2+PI**2)/2.
11162      S2=0.875/(SI**2+PI**2)
11158      S3=1.00/(4.*SI**2+9.*PI**2)
11168      S4=1./(32.*(SI**2+4.*PI**2))
11178      C=4.
11182      D=12.
11184      E=20.25
11186      F=20.5
11187      G=12.71875
11189      H=4.46875
11191      P=0.6982422
11196      TA=.54654*50.5*(C+A*(D+A*(E+A*(F+A*(G+A*(H+P*A))))))
11204      EXTRA=1.09375+.2914*SI*(S1-S2+S3-S4)
11206      U04=.25*(AH**2)*PI*SI*(SA*EXTRA+TA)
11208      PRINT 51,U04
11208      51 FORMAT(1IH MU(0,4)=E15.8)
11234      C=1.
11258      D=3.
11282      E=75./16.
11288      F=35./8.
11294      G=75./128.
11296      H=99./128.
11298      P=29./4096.
11262      RA=55.*SI*PI*(A**2)*(C+A*(D+A*(E+A*(F+A*(G+A*(H+P*A))))))
11308      TT=0.3125*(PI**3/3/SI)*SA+RA
11322      PRINT 52,TT
11346      52 FORMAT(1IH MU(2,4)=E15.8)
11358      T=TT/U04
11364      PRINT 1,T
11366      END

```

Program 2.

This program is divided into two parts. Part I is a program to compute far-field powers of

$G_1(u, o)$  [page 9] and  $G_2(u, o)$  [page 12]

The increment of  $u$  is approximately 0.01.  
Part II, a similar program, computes

$G_3(u, o)$  [page 15] and  $G_4(u, o)$  [page 16]

The value of  $a$  is chosen to be 99.

Part I

```

07300 3 FORMAT(18H VOLTAGE-POWER 1/)
07366 4 FORMAT(18H VOLTAGE-POWER 2/)
07432 5 FORMAT(F10.4)
07454 7 FORMAT(12H G1(0,0)=E15.8)
07508 8 FORMAT(16H LCG G1(0,0)=E15.8)
07570 10 FORMAT(4H F10.5)
07608 11 FORMAT(3H //)
07654 12 FORMAT(10H G(0,0)=E15.8)
07704 17 FORMAT(12H G2(0,0)=E15.8)
07758 18 FORMAT(16H LCG G2(0,0)=E15.8)
07820 DIMENSION Y0(60),Y1(60),Y2(60),Y3(60),Y4(60),Y5(60),Y6(60)
07820 DC 6 I=1,60
07832 6 READ 5, Y0(I)
07916 DC 70 J=1,60
07928 70 READ 5, Y1(J)
08012 PRINT 3
08036 A=99.
08072 PI=22./7.
08120 S1=0.277
08156 B=PI*(1.+S1**2)/(16.*S1**2+PI**2)
08336 A0=49.5+101.0*PI*B
08408 A2=101.0+A*X*PI*B
08480 A4=49.5
08516 PRINT 7,A0
08540 IF(A0) 37,38,38
08596 37 A0--A0
08644 38 C=LCGF(A0)
08680 PRINT 8,C
08704 I=0
08740 AX=0.0
08776 100 I=I+1
08824 AX=AX+.25
08872 Y2(I)=(2./AX)*Y1(I)-Y0(I)
09016 Y3(I)=(4./AX)*Y2(I)-Y1(I)
0916C Y4(I)=(6./AX)*Y3(I)-Y2(I)
09304 POWER=A0*Y0(I)+A2*Y2(I)+A4*Y4(I)
09520 IF(POWER) 57,58,58
09576 57 POWER=POWER
09624 58 P=LCGF(POWER)
09660 POWER=8.6858*(P-C)
09720 PRINT 10,POWER
09744 IF(I=60) 100,108,108
09812 108 PRINT 11
09836 PRINT 4
09860 P=PI**2
09908 Q=S1**2
09956 B=0.5*P/(4.*Q+P)
10064 A0=(1.+A**1.+375*A)**B+.25*A*(1.+5*A)
10268 PRINT 17,A0
10292 IF(A0) 97,98,98
10348 97 A0--A0
10396 98 T=LCGF(A0)
10432 PRINT 18,T
10456 A2=(1.+.5*A)*A*B+.25*(2.+A*(2.+7.*A/8.))
10648 A4=.125*A*(A*B+2.*(.1+.5*A))
10816 A6=(A**2)/32.
10876 I=0
10912 AX=0.0
10949 200 I=I+1
10956 AX=AX+.25
11044 Y2(I)=(2./AX)*Y1(I)-Y0(I)
11188 Y3(I)=(4./AX)*Y2(I)-Y1(I)
11332 Y4(I)=(6./AX)*Y3(I)-Y2(I)
11476 Y5(I)=(8./AX)*Y4(I)-Y3(I)
11620 Y6(I)=(10./AX)*Y5(I)-Y4(I)
11764 POWER=A0*Y0(I)+A2*Y2(I)+A4*Y4(I)+A6*Y6(I)
12052 IF(POWER) 87,88,88
12108 87 POWER=POWER
12156 88 P=LCGF(POWER)
12192 POWER=8.6858*(P-T)
12252 PRINT 10,POWER
12276 IF(I=60) 200,208,208
12344 208 PAUSE
12356 END

```

## Part II

```

07300      5 FORMAT(18H VOLTAGE-POWER 3/)
07366      6 FORMAT(18H VOLTAGE-POWER 4/)
07432      13 FORMAT(13H LN G(0,0)=E15.8)
07488      15 FORMAT(15H F10.5/)
07560      16 FORMAT(11H G4(0,0)=E15.8)
07612      17 FORMAT(F10.4)
07634      DIMENSION Y0(60),Y1(60),Y2(60),Y3(60),Y4(60),Y5(60),Y6(60)
07634      DIMENSION Y7(60),Y8(60),Y9(60),Y10(60)
A=99.
P1=22./7.
S1=.277
P=P1**2
Q=S1**2
07802      D0 1 I=1,60
07850      1 READ 17, Y0(I)
07862      D0 2 J=1,60
07946      2 READ 17, Y1(J)
08042      PRINT 5
08066      B1=1./(16.*Q+P)
08138      B2=1./(16.*Q+9.*P)
08246      B=B1-B2
08294      PCNE=1.+A*(1.5+A*(1.125+A*0.3125))
08402      PTWC=A*(1.5+A*(1.5+15.*A/32.))
08510      SQ=1.+(.277*.277)
08570      A0=1.5**SQ*B*PCNE+2.*PTWC/3.
08714      A2=1.5**P*SQ*B*PTWC+2.*((2.+A*(3.+A*(21./8.+13.*A/16.)))/3.
08978      A4=.5625*P*SQ*.5*(A**2)*B+2.*A*(1.5+A*(1.5+.5*A))/3.
09254      A6=(3./64.)*P*SQ*(A**3)*B+.25*50.5*A*A
09458      A8=A**3/48.
09518      A10=0.0
09554      IF(A0) 37,38,38
09610      37 A0=A0
09658      38 POWER=LOGF(A0)
09694      PRINT 13,POWER
09718      J=0
09754      200 I=0
09790      AX=0.0
09826      J=I+1
09874      100 I=I+1
09922      AX=AX+.25
09970      Y2(I)=2./AX)*Y1(I)-Y0(I)
T0114      Y3(I)=4./AX)*Y2(I)-Y1(I)
T0253      Y4(I)=(6./AX)*Y3(I)-Y2(I)
T0402      Y5(I)=(8./AX)*Y4(I)-Y3(I)
T0546      Y6(I)=(10./AX)*Y5(I)-Y4(I)
T0690      Y7(I)=(12./AX)*Y6(I)-Y5(I)
T0834      Y8(I)=(14./AX)*Y7(I)-Y6(I)
T0978      Y9(I)=(16./AX)*Y8(I)-Y7(I)
T1122      Y10(I)=(18./AX)*Y9(I)-Y8(I)
T1266      G=A0*Y0(I)+A2*Y2(I)+A4*Y4(I)+A6*Y6(I)+A8*Y8(I)+A10*Y10(I)
T1698      IF(G) 57,58,58
T1754      57 G=-G
T1802      58 GG=LOGF(G)
T1838      R=.6858*(GG-POWER)
T1898      PRINT 15,R
T1922      IF(I=60) 100,206,206
T1990      206 IF(J=2) 207,208,208
T2058      208 PAUSE
T2070      207 PRINT 6
T2094      SS=(.277*.277)*(2./(4.*Q+P)-.125/(P+0))
T2286      C=35./128.
T2334      TTT=(1.+A*(2.+A*(2.25+A*(1.25+C*A))))*(.375-SS)
T2514      A0=TTT+.1875*A*(2.+A*(3.+A*(1.875+.7.*A/16.)))
T2694      PRINT 16,A0
T2718      IF(A0) 67,68,68
T2774      67 A0=A0
T2822      68 POWER=LOGF(A0)
T2858      PRINT 13,POWER
T2882      C=5.25
T2918      D=3.25
T2954      E=49./64.
T3002      T=0.1375*(2.+A*(4.+A*(C+A*(D+A*E))))
T3146      A2=A*(2.+A*(3.+A*(1.875+0.4375*A)))*(0.375-SS)+T
T3326      C=7./32.
T3374      R=(A**2)*(0.75+A*(0.75+A*C))
T3506      T=0.1875*A*(2.+A*(3.+A*(2.+0.5*A)))
T3662      A4=R*(0.375-SS)+T
T3734      R=0.125*(A**3)*(1.+0.5*A)
T3854      T=(A*A)*0.1875*(0.75+A*(0.75+29.*A/128.))
T4010      A6=R*(0.375-SS)+T
T4082      A8=(A**4/128.)*(0.375-SS)+(3./128.)*A**3*(1.+0.5*A)
T4446      A10=3.*A**4/2048.
T4426      GO TO 200
END

```